Five Year Integrated M. Sc. Examination, 2021

Semester-V

Course: MT-3-5-2

(Analysis I)

Time: Four Hours **Full Marks: 80**

Questions are of values as indicated in the margin.
Notations and symbols have their usual meanings.

Answer ***any four*** questions.

1. (a) Let $\{f_n\}$ be a sequence of functions on $\mathbb{R}$. Explain the difference between the pointwise convergence and uniform convergence of $\{f_n\}$.
If $\{f_n\}$ converges uniformly to a function f, then show that $\{f_n\}$ converges pointwise to f. Does the converse hold? Justify your answer. [2+4+4]
(b) Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a uniformly continuous function on $\mathbb{R}$ and suppose that $f_n(x) = f(x + c_n)$, $x \in \mathbb{R}$ for $n = 1, 2, \ldots$, where $\{c_n\}$ is a sequence converging to 0. Show that $\{f_n\}$ is uniformly convergent on $\mathbb{R}$. [6]
(c) Exhibit by considering an example that although a sequence $\{f_n\}$ of integrable functions converges pointwise to a function f on $[a, b]$, the function f may be integrable on $[a, b]$ and $\lim\limits_{n\to\infty} \int_a^b f_n = \int_a^b f$ may hold. [4]

2. (a) Let the function $f_n : [a, b] \longrightarrow \mathbb{R}$ be defined for $n = 1, 2, \ldots$ and suppose that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ and the series $\sum f_n$ is uniformly convergent on $[x_{r-1}, x_r]$ for $r = 1, 2, \ldots, n$. Prove that $\sum f_n$ is uniformly convergent on $[a, b]$. [5]
(b) Let $\sum b_n$ be an absolutely convergent series of real numbers. Show that the series $\sum \dfrac{b_n x^n}{1 + x^{2n}}$ converges absolutely and uniformly for all real x. Also discuss the absolute and uniform convergence of $\sum b_{2n} \sin n^2 x$ for all real x. [5+5]
(c) Use Abel's test to determine the uniform convergence of the series $\sum \dfrac{a_n}{n^x}$, provided that $\sum a_n$ is a convergent series of real numbers. [5]

3. (a) Determine the values of n for which the improper integral $\displaystyle\int_a^b \frac{dx}{(b - x)^n}$ converges at b. [5]
(b) Test the convergence of $\displaystyle\int_0^\pi \frac{\sqrt{x}}{\sin x} dx$ and $\displaystyle\int_0^\infty x^3 e^{-x^2} dx$. [5+5]
(c) Does $\displaystyle\int_0^\infty \frac{\sin x}{x} dx$ converge absolutely? Justify. [5]

4. (a) Find the values of m and n such that $\displaystyle\int_0^1 x^{m-1}(1 - x)^{n-1} dx$ is convergent. [6]
(b) Prove or disprove: The sum function of a power series $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ is continuous on its interval of convergence. [6]
(c) Find the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} (5^n + 6^n) x^n$. Also find the sum of the series and the domain of definition of the sum. [4+4]

5. (a) Let $f : [a,b] \longrightarrow \mathbb{R}$ be a bounded function on $[a,b]$ and P_1, P_2 be any two partitions of $[a,b]$. Show that $L(P_1, f) \leq U(P_2, f)$. Also deduce that $\underline{\int_a^b} f \leq \overline{\int_a^b} f dx$. [4+4]

(b) A function $f : [0,3] \longrightarrow \mathbb{R}$ is defined by $f(x) = x[x]$. Show that f is Riemann integrable on $[0,3]$ and evaluate $\int_0^3 f$. [3+3]

(c) If $f : [a,b] \longrightarrow \mathbb{R}$ is Riemann integrable on $[a,b]$, then show that the function g defined by $g(x) = \int_a^x f(y)dy$ is continuous on $[a,b]$. Does the derivative g' of g exist on $[a,b]$. Justify. [3+3]

6. (a) Express the limit $\lim_{n\to\infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+3n}\right]$ as the Riemann integral of a function on an interval and find its value. [4+2]

(b) State the Second Mean Value Theorem (Weierstrass form). Verify it for the function $f(x) = xe^x$ on $[-1,1]$. [2+4]

(c) Let $f_n : [a,b] \longrightarrow \mathbb{R}$ be a continuous function on $[a,b]$ for each $n = 1, 2, \ldots$ such that $\{f_n\}$ is uniformly convergent on $[a,b]$. Define $g_n(x) = \int_a^x f_n(x)dx$, for $a \leq x \leq b$ and for each $n = 1, 2, \ldots$. Prove that $\{g_n\}$ is uniformly convergent on $[a,b]$. [4]

(d) State Fundamental Theorem of Integral Calculus. Use it to evaluate $\int_{-3}^{3} f$, where

$$f(x) = \begin{cases} 2x \sin\frac{\pi}{x} - \pi \cos\frac{\pi}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

. [2+2]